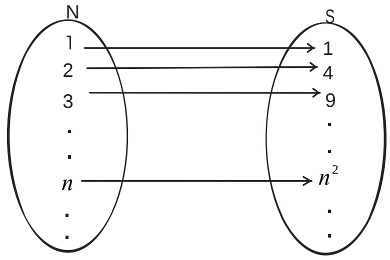
Sequence and Series

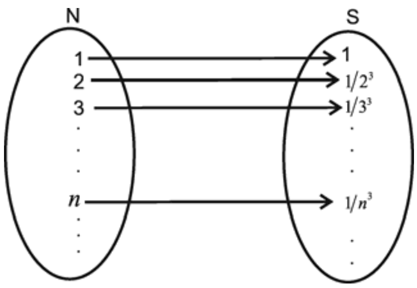
Sequence

A function f:N $\rightarrow$ S, where S is any nonempty set is called a *Sequence* i.e., for each n$\in$ N, $\exists$ a unique element f(n) $\in$ S. The sequence is written as f(1), f(2), f(3), ...f(n)...., and is denoted by {f(n)}, or <f(n)>, or (f(n)). If f(n) = a_n , the sequence is written as $a_1, a_2 .....a_n$ and denoted by , $\{a_n\}$ $or < a_n > or \left(a_n\right)$. Here f(n) or a_n are the n^{th} terms of the Sequence.

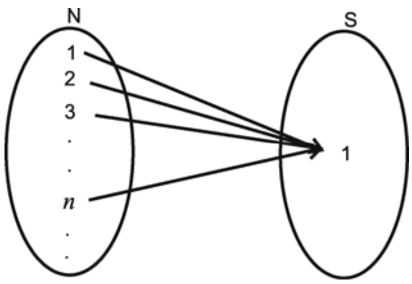
***Ex*. 1.** 1 , 4 , 9 , 16 ,......... n^2 ,.....(or) $< n^2 >$



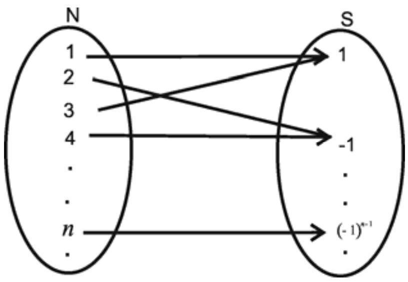
***Ex*. 2.** $\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, ...\frac{1}{n^3}....(or)\left(\frac{1}{n^3}\right)$



***Ex*. 3.** 1, 1, 1......1..... or <1>



Ex 4: 1 , –1, 1, –1, ... or $\langle (-1)^{n-1} \rangle$



Note : 1. If S $\subseteq$ R then the sequence is called a *real sequence.*
2. The range of a sequence is almost a countable set.

Kinds of Sequences

1. Finite Sequence: A sequence $< a_n >$ in which $a_n = 0 \ \forall n > m \in N$ is said to be a finite Sequence. i.e., A finite Sequence has a finite number of terms.

2. Infinite Sequence: A sequence, which is not finite, is an infinite sequence.

Bounds of a Sequence and Bounded Sequence

1. If $\exists$ a number 'M' $\ni a_n \leq$ M, $\forall n \in$ N, the Sequence $< a_n >$ is said to be bounded above or bounded on the right.

Ex. $1, \frac{1}{2}, \frac{1}{3},$,....... here $a_n \leq 1 \ \forall n \in$ N

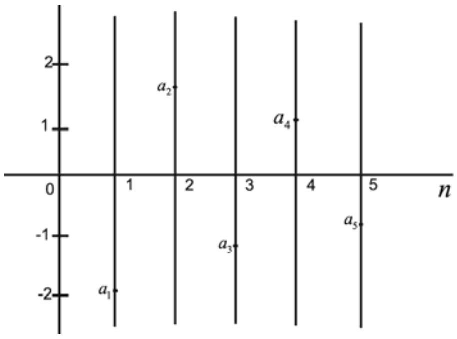
2. If $\exists$ a number 'm' $\ni a_n \geq m, \forall n \in$ N, the sequence $< a_n >$ is said to be bounded below or bounded on the left.

Ex. 1 , 2 , 3 ,.....here $a_n \geq 1 \ \forall n \in$ N

3. A sequence which is bounded above and below is said to be bounded.

Ex. Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

n	1	2	3	4	......
a_n	-2	3/2	-4/3	5/4	......



From the above figure (see also table) it can be seen that m = –2 and M = $\frac{3}{2}$.

$\therefore$ The sequence is bounded.

Limits of a Sequence

A Sequence $< a_n >$ is said to tend to limit 'l' when, given any + ve number '$\in$', however small, we can always find an integer 'm' such that $|a_n - l| < \in, \forall n \geq m$, and we write $\underset{n\to\infty}{Lt}\ a_n = l$ or $\langle a_n \to l \rangle$

Ex. If $a_n = \frac{n^2+1}{2n^2+3}$ then $< a_n > \to \frac{1}{2}$.

Convergent, Divergent and Oscillatory Sequences

1. **Convergent Sequence:** A sequence which tends to a finite limit, say 'l' is called a *Convergent Sequence.* We say that the sequence converges to 'l'
2. **Divergent Sequence:** A sequence which tends to $\pm\infty$ is said to be *Divergent* (or is said to diverge).
3. **Oscillatory Sequence:** A sequence which neither converges nor diverges ,is called an *Oscillatory Sequence.*

***Ex.* 1.** Consider the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...$ here $a_n = 1 + \frac{1}{n}$

The sequence $< a_n >$ is convergent and has the limit 1

$a_n - 1 = 1 + \frac{1}{n} - 1 = \frac{1}{n}$ and $\frac{1}{n} < \in$ whenever $n > \frac{1}{\in}$

Suppose we choose $\in = .001$, we have $\frac{1}{n} < .001$ when n > 1000.

***Ex.* 2.** If $a_n = 3 + (-1)^n \frac{1}{'n'} < a_n >$ converges to 3.

Ex. 3. If $a_n = n^2 + (-1)^n .n, < a_n >$ diverges.

Ex. 4. If $a_n = \frac{1}{n} + 2(-1)^n, < a_n >$ oscillates between -2 and 2.

Infinite Series

If $< u_n >$ is a sequence, then the expression $u_1 + u_2 + u_3 + ... + u_n + ...$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$ or simply Σu_n

The sum of the first *n* terms of the series is denoted by s_n

i.e., $s_n = u_1 + u_2 + u_3 + ... + u_n; s_1, s_2, s_3, ...s_n$ are called *partial sums.*

Convergent, Divergent and Oscillatory Series

Let Σu_n be an infinite series. As $n \to \infty,$ there are three possibilities.

(a) **Convergent series:** As $n \to \infty, s_n \to$ a finite limit, say 's' in which case the series is said to be convergent and 's' is called its sum to infinity.

Thus $\underset{n \to \infty}{Lt}\, s_n = s$ (or) simply $Lt s_n = s$

This is also written as $u_1 + u_2 + u_3 + ... + u_n + ...to\infty = s.$ (or) $\sum_{n=1}^{\infty} u_n = s$ (or) simply $\Sigma u_n = s.$

(b) **Divergent series:** If $s_n \to \infty$ or $-\infty$, the series said to be divergent.

(c) **Oscillatory Series:** If s_n does not tend to a unique limit either finite or infinite it is said to be an *Oscillatory Series.*

Note: Divergent or Oscillatory series are sometimes called non convergent series.

Geometric Series

The series, $1 + x + x^2 + ...x^{n-1} + ...$ is

(i) Convergent when $|x| < 1$, and its sum is $\frac{1}{1-x}$

(ii) Divergent when $x \geq 1$.

(iii) Oscillates finitely when x = -1 and oscillates infinitely when *x* < -1.

Proof: The given series is a geometric series with common ratio '*x*'

$\therefore\ s_n = \frac{1-x^n}{1-x}$ when $x \neq 1$ [By actual division – verify]

(i) When $|x| < 1$:

$$\underset{n\to\infty}{Lt}\, s_n = \underset{n\to\infty}{Lt}\left(\frac{1}{1-x}\right) - \underset{n\to\infty}{Lt}\left(\frac{x^n}{1-x}\right) = \frac{1}{1-x} \qquad \left[\text{since } x^n \to 0 \text{ as } n \to \infty\right]$$

$\therefore$ The series converges to $\frac{1}{1-x}$

(ii) When $x \geq 1: s_n = \frac{x^n - 1}{x-1}$ and $s_n \to \infty$ as $n \to \infty$

$\therefore$ The series is divergent.

(iii) When $x = -1$: when n is even, $s_n \to 0$ and when n is odd, $s_n \to 1$

$\therefore$ The series oscillates finitely.

(iv) When $x < -1, s_n \to \infty$ or $-\infty$ according as n is odd or even.

$\therefore$ The series oscillates infinitely.

Some Elementary Properties of Infinite Series

1. The convergence or divergence of an infinites series is unaltered by an addition or deletion of a finite number of terms from it.

2. If some or all the terms of a convergent series of positive terms change their signs, the series will still be convergent.

3. Let Σu_n converge to 's'

Let 'k' be a non – zero fixed number. Then Σku_n converges to ks.

Also, if Σu_n diverges or oscillates, so does Σku_n

4. Let Σu_n converge to 'l' and Σv_n converge to 'm'. Then

(i) $\Sigma(u_n + v_n)$ converges to ($l + m$) and **(ii)** $\Sigma(u_n + v_n)$ converges to ($l - m$)

Series of Positive Terms

Consider the series in which all terms beginning from a particular term are +ve.

Let the first term from which all terms are +ve be u_1.

Let Σu_n be such a convergent series of +ve terms. Then, we observe that the convergence is unaltered by any rearrangement of the terms of the series.

Theorem

If Σu_n is convergent, then $\underset{n\to\infty}{Lt}\, u_n = 0$.

Proof : $s_n = u_1 + u_2 + ... + u_n$

$s_{n-1} = u_1 + u_2 + ... + u_{n-1,}$, so that, $u_n = s_n - s_{n-1}$

Suppose $\Sigma u_n = l$ then $\underset{n\to\infty}{Lt}\, s_n = l$ and $\underset{n\to\infty}{Lt}\, s_{n-1} = l$

$\therefore \underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\left(s_n - s_{n-1}\right)$; $\underset{n\to\infty}{Lt}\, s_n - \underset{n\to\infty}{Lt}\, s_{n-1} = l - l = 0$

Note: The converse of the above theorem need not be always true. This can be observed from the following examples.

(i) Consider the series, $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$; $u_n = \frac{1}{n}, \underset{n\to\infty}{Lt}\, u_n = 0$

But from p-series test (1.3.1) it is clear that $\Sigma \frac{1}{n}$ is divergent.

(ii) Consider the series, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} + ...$

$u_n = \frac{1}{n^2}, \underset{n\to\infty}{Lt}\, u_n = 0,$ by p series test, clearly $\Sigma \frac{1}{n^2}$ converges,

Note : If $\underset{n\to\infty}{Lt}\, u_n \neq 0$ the series is divergent;

Ex. $u_n = \frac{2^n - 1}{2^n}$, here $\underset{n\to\infty}{Lt}\, u_n = 1 \quad \therefore \quad \Sigma u_n$ is divergent.

Tests for the Convergence of an Infinite Series

In order to study the nature of any given infinite series of +ve terms regarding convergence or otherwise, a few tests are given below.

P-Series Test

The infinite series, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ...,$ is

(i) Convergent when $p > 1$, and **(ii)** Divergent when $p \leq 1$. (JNTU 2002, 2003)

Proof :

***Case* (i)** Let $p > 1$; $p > 1, 3^p > 2^p$; $\Rightarrow \frac{1}{3^p} < \frac{1}{2^p}$

$$\therefore \quad \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p}$$

Similarly, $$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p}$$

$$\frac{1}{8^p} + \frac{1}{9^p} + ... + \frac{1}{16^p} < \frac{8}{8^p},$$ and so on.

Adding we get

$$\Sigma\frac{1}{n^p} < 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+....$$

i.e., $$\Sigma\frac{1}{n^p} < 1+\frac{1}{2^{(p-1)}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+......$$

The RHS of the above inequality is an infinite geometric series with common ratio $\frac{1}{2^{p-1}} < 1(\text{since } p > 1)$ The sum of this geometric series is finite.

Hence $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is also finite.

$\therefore$ The given series is convergent.

***Case* (ii)** Let p =1; $\Sigma\frac{1}{n^p} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+......$

We have, $$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9}+\frac{1}{10}+.......\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+.....\frac{1}{16} = \frac{1}{2}$$ and so on

$\therefore$ $$\Sigma\frac{1}{n^p} = 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+.....$$

$$\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....$$

The sum of RHS series is ∞

$$\left(\text{since } s_n = 1+\frac{n-1}{2} = \frac{n+1}{2} \text{ and } \underset{n\to\infty}{Lt}\, s_n = \infty\right)$$

$\therefore$ The sum of the given series is also ∞; $\therefore$ $\sum_{n=1}^{\infty}\frac{1}{n^p}$ (p = 1) diverges.

Case (iii) Let p<1, $\Sigma\frac{1}{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+.....$

Since $$p < 1, \frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}......$$ and so on

$\therefore$ $$\Sigma\frac{1}{n^p} > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......$$

From the Case (ii), it follows that the series on the RHS of above inequality is divergent.

$\therefore \qquad \Sigma \frac{1}{n^p}$ is divergent , when P < 1

Note: *This theorem is often helpful in discussing the nature of a given infinite series.*

Comparison Tests

1. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be convergent.

Then Σu_n converges,

(a) If $u_n \le v_n, \forall \ n \in N$

(b) or $\frac{u_n}{v_n} \le k \forall n \in N$ where k is > 0 and finite.

(c) or $\frac{u_n}{v_n} \to$ a finite limit > 0

Proof : (a) Let $\Sigma v_n = l$ (finite)

Then, $u_1 + u_2 + ... + u_n + ... \le v_1 + v_2 + ... v_n + ... \le l > 0$

Since l is finite it follows that Σu_n is convergent

(c) $\frac{u_n}{v_n} \le k \Rightarrow u_n \le k v_n, \forall n \in N$, since Σv_n is convergent and k (>0) is finite, $\Sigma k v_n$ is convergent $\therefore$ Σu_n is convergent.

(d) Since $\underset{n \to \infty}{Lt} \frac{u_n}{v_n}$ is finite, we can find a +ve constant $k, \ni \frac{u_n}{v_n} < k \forall n \in N$

$\therefore$ from (2) , it follows that Σu_n is convergent

2. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be divergent. Then Σu_n diverges,

* 1. If $u_n \ge v_n, \forall n \in N$

or * 2. If $\frac{u_n}{v_n} \ge k, \forall n \in N$ where k is finite and $\ne 0$

or * 3. If $\underset{n \to \infty}{Lt} \frac{u_n}{v_n}$ is finite and non-zero.

Proof:

1. Let M be a +ve integer however large it may be. Science Σv_n is divergent, a number m can be found such that

$$v_1 + v_2 + ... + v_n > M, \forall n > m$$

$\therefore \qquad u_1 + u_2 + ... + u_n > M, \forall n > m \left(u_n \ge v_n\right)$

$\therefore \qquad \Sigma u_n$ is divergent

2. $u_1 \geq kv_n \forall n$

Σv_n is divergent $\Rightarrow$ Σkv_n is divergent

$\therefore$ Σu_n is divergent

3. Since $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}$ is finite, a + ve constant k can be found such that $\dfrac{u_n}{v_n} > k, \forall n$ (probably except for a finite number of terms)

$\therefore$ From (2), it follows that Σu_n is divergent.

Note :

(a) In (1) and (2), it is sufficient that the conditions with * hold $\forall n > m \in N$

Alternate form of comparison tests : The above two types of comparison tests 2.8.(1) and 2.8.(2) can be clubbed together and stated as follows :

If Σu_n and Σv_n are two series of + ve terms such that $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = k$, where k is non- zero and finite, then Σu_n and Σv_n both converge or both diverge.

(b)
1. The above form of comparison tests is mostly used in solving problems.
2. In order to apply the test in problems, we require a certain series Σv_n whose nature is already known i.e., we must know whether Σv_n is convergent are divergent. For this reason, we call Σv_n as an 'auxiliary series'.
3. In problems, the geometric series (1.2.2.) and the *p*-series (1.3.1) can be conveniently used as 'auxiliary series'.

Solved Examples

EXAMPLE 1

Test the convergence of the following series:

(a) $\dfrac{3}{1}+\dfrac{4}{8}+\dfrac{5}{27}+\dfrac{6}{64}+.....$ **(b)** $\dfrac{4}{1}+\dfrac{5}{4}+\dfrac{6}{9}+\dfrac{7}{16}+.....$ **(c)** $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]$

SOLUTION

(a) **Step 1:** To find "u_n" the n^{th} term of the given series. The numerators 3, 4, 5, 6......of the terms, are in AP.

n^{th} term $t_n = 3+\left(n-1\right).1 = n+2$

Denominators are $1^3, 2^3, 3^3, 4^3 .....n^{th}$ term = n^3 ; $\therefore$ $u_n = \dfrac{n+2}{n^3}$

Step 2: To choose the auxiliary series Σv_n. In u_n, the highest degree of n in the numerator is 1 and that of denominator is 3.

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$\therefore$ we take, $\quad v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$

Step 3: $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n+2}{n^3} \times n^2 = \underset{n\to\infty}{Lt} \frac{n+2}{n} = \underset{n\to\infty}{Lt} \left(1+\frac{2}{n}\right) = 1,$ which is non- zero and finite.

Step 4: ***Conclusion:*** $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = 1$

$\therefore$ Σu_n and Σv_n both converge or diverge (by comparison test). But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent by *p*-series test ($p = 2 > 1$); $\therefore$ Σu_n is convergent.

(b) $\frac{4}{1} + \frac{5}{4} + \frac{6}{9} + \frac{7}{16} + ...$

Step 1: 4 , 5, 6, 7, ...in AP , $t_n = 4 + (n-1)1 = n+3 \qquad \therefore \ u_n = \frac{n+3}{n^2}$

Step 2: Let $\Sigma v_n = \frac{1}{n}$ be the auxiliary series

Step 3: $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \left(\frac{n+3}{n^2}\right) \times n = \underset{n\to\infty}{Lt} \left(1+\frac{3}{n}\right) = 1$, which is non-zero and finite.

Step 4: $\therefore$ By comparison test, both Σu_n and Σv_n converge are diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent, by *p*-series test (p = 1); $\therefore$ Σu_n is divergent.

(c) $\sum_{n=1}^{\infty} \left[\left(n^4+1\right)^{1/4} - n \right] = \left\{ n^4 \left(1+\frac{1}{n^4}\right) \right\}^{\frac{1}{4}} - n = n\left[\left(1+\frac{1}{n^4}\right)^{\frac{1}{4}} - 1 \right]$

$$= n\left[1 + \frac{1}{4n^4} + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!} . \frac{1}{n^8} + ... - 1 \right] = n\left[\frac{1}{4n^4} - \frac{3}{32n^8} + ... \right]$$

$$= \frac{1}{4n^3} - \frac{3}{32n^7} + ... = \frac{1}{n^3}\left[\frac{1}{4} - \frac{3}{32n^4} + ... \right]$$

Here it will be convenient if we take $v_n = \frac{1}{n^3}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\left(\frac{1}{4}-\frac{1}{32n^4}+.....\right)=\frac{1}{4},$$ which is non-zero and finite

$\therefore$ By comparison test, Σu_n and Σv_n both converge or both diverge. But by *p*-series test $\Sigma v_n=\frac{1}{n^3}$ is convergent. (p = 3 > 1); $\therefore$ Σu_n is convergent.

EXAMPLE 2

If $u_n=\frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{2n^3+3n+5}}$ show that Σu_n is divergent

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SOLUTION

As n increases, u_n approximates to

$$\frac{\sqrt[3]{3n^2}}{\sqrt[4]{2n^3}}=\frac{3^{1/3}}{2^{1/4}}\times\frac{n^{2/3}}{n^{3/4}}=\frac{3^{1/3}}{2^{1/4}}.\frac{1}{n^{1/12}}$$

$\therefore$ If we take $v_n=\frac{1}{n^{1/12}}$, $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{3^{1/3}}{2^{1/4}}$ which is finite.

[(or) *Hint:* Take $v_n=\frac{1}{n^{l_1-l_2}}$, where l_1 and l_2 are indices of ‘n’ of the largest terms in denominator and nominator respectively of u_n. Here $v_n=\frac{1}{n^{\frac{3}{4}-\frac{2}{3}}}=\frac{1}{n^{\frac{1}{12}}}$]

By comparison test, Σv_n and Σu_n converge or diverge together. But $\Sigma v_n=\Sigma\frac{1}{n^{1/12}}$ is divergent by p – series test (since $p=\frac{1}{12}<1$)

$\therefore$ Σu_n is divergent.

EXAMPLE 3

Test for the convergence of the series. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\sqrt{\frac{4}{5}}+......$

SOLUTION

Here, $u_n=\sqrt{\frac{n}{n+1}}$; Take $v_n=\frac{1}{n^{\frac{1}{2}-\frac{1}{2}}}=\frac{1}{n^0}=1$, $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\sqrt{\frac{1}{1+\frac{1}{n}}}=1$ (finite)

Σv_n is divergent by p – series test. ($p = 0 < 1$)

$\therefore$ By comparison test, Σu_n is divergent, (Students are advised to follow the procedure given in ex. 1.2.9(a) and (b) to find "u_n" of the given series.)

EXAMPLE 4

Show that $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+.......+\frac{1}{\lfloor n}+.....$ is convergent.

SOLUTION

$$u_n = \frac{1}{\lfloor n} \quad \text{(neglecting 1}^{\text{st}}\text{ term)}$$

$$= \frac{1}{1.2.3......n} < \frac{1}{1.2.2.2.....\overline{n-1}\,times} = \frac{1}{(2^{n-1})}$$

$\therefore$ $$\Sigma u_n < 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+......$$

which is an infinite geometric series with common ratio $\frac{1}{2}<1$

$\therefore$ $\Sigma\frac{1}{2^{n-1}}$ is convergent. (1.2.3(a)). Hence Σu_n is convergent.

EXAMPLE 5

Test for the convergence of the series, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+.......$

SOLUTION

$$u_n = \frac{1}{n(n+1)(n+2)}; \qquad \text{Take } v_n = \frac{1}{n^3} \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{n^3}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 \text{ (finite)}$$

$\therefore$ By comparison test, Σu_n, and Σv_n converge or diverge together. But by p-series test, $\Sigma v_n = \Sigma\frac{1}{n^3}$ is convergent ($p = 3 > 1$); $\therefore$ Σu_n is convergent .

EXAMPLE 6

If $u_n = \sqrt{n^4+1}-\sqrt{n^4-1}$, show that Σu_n is convergent. [JNTU, 2005]

SOLUTION

$$u_n = n^2\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} - n^2\left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}$$

$$= n^2\left[\left(1+\frac{1}{2n^4}-\frac{1}{8n^8}+\frac{1}{16n^{12}}-....\right)-\left(1-\frac{1}{2n^4}-\frac{1}{8n^8}-\frac{1}{16n^{12}}-.....\right)\right]$$

$$= n^2\left[\frac{1}{n^4}+\frac{1}{8n^{12}}+....\right]=\frac{1}{n^2}\left[1+\frac{1}{8n^{10}}+....\right]$$

Take $v_n = \frac{1}{n^2}$, hence $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1$

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \frac{1}{n^2}$ is convergent by p –series test (p = 2 > 1) $\therefore$ Σu_n is convergent.

EXAMPLE 7

Test the series $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+.....$ for convergence.

SOLUTION

$$u_n = \frac{1}{n+x}; \quad \text{take} \quad v_n = \frac{1}{n}, \quad \text{then} \quad \frac{u_n}{v_n}=\frac{n}{n+x}=\frac{1}{1+\frac{x}{n}}$$

$$\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{x}{n}}\right)=1; \Sigma v_n = \Sigma\frac{1}{n}$$ is divergent by p-series test (p =1)

$\therefore$ By comparison test, Σu_n is divergent.

EXAMPLE 8

Show that $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ is divergent.

SOLUTION

$$u_n = \sin\left(\frac{1}{n}\right); \quad \text{take} \quad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \underset{t\to 0}{Lt}\frac{\sin t}{t} \ \left(\text{where } t = 1/n\right) = 1$$

$\therefore$ $\Sigma u_n , \Sigma v_n$ both converge or diverge . But $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent

(p -series test, $p = 1$); $\therefore$ Σu_n is divergent.

EXAMPLE 9

Test the series $\Sigma \sin^{-1}\left(\frac{1}{n}\right)$ for convergence.

SOLUTION

$$u_n = \sin^{-1}\frac{1}{n}; \qquad \text{Take} \qquad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin^{-1\left(\frac{1}{n}\right)}}{\left(\frac{1}{n}\right)}; = \underset{\theta\to 0}{Lt}\left(\frac{\theta}{\sin\theta}\right) = 1\left(Taking\ \sin^{-1}\frac{1}{n} = \theta\right)$$

But Σv_n is divergent. Hence Σu_n is divergent.

EXAMPLE 10

Show that the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^3} + ...$ is divergent.

SOLUTION

Neglecting the first term, the series is $\frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + ...$. Therefore

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)}n = \frac{n^n}{n\left(1+\frac{1}{n}\right).n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n};$$

Take $v_n = \frac{1}{n}$

$$\therefore \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^n.1} = \frac{1}{e}$$

which is finite and $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent by p –series test (p = 1)

$\therefore \quad \Sigma u_n$ is divergent.

EXAMPLE 11

Show that the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$ is convergent. (JNTU 2000)

SOLUTION

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$$

n^{th} term = $u_n = \dfrac{2n-1}{n(n+1)(n+2)} = \dfrac{1}{n^2}\cdot\dfrac{\left(2-\dfrac{1}{n}\right)}{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)}$

Take $v_n = \dfrac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^2}\frac{\left(2-\dfrac{1}{n}\right)}{\left(1+{}^{1}\!/\!_{n}\right)\left(1+{}^{2}\!/\!_{n}\right)} \div \left(\frac{1}{n^2}\right)$$

$\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \dfrac{2-0}{(1+0)(1+0)} = 2$ which is finite and non-zero

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \dfrac{1}{n^2}$ is convergent. $\therefore$ $\sum u_n$ is also convergent.

EXAMPLE 12

Test whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ is convergent (JNTU 1997, 1999, 2003)

SOLUTION

The given series is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}-\sqrt{n}\right)} = \sqrt{n+1}-\sqrt{n}$$

$$u_n = \sqrt{n}\left\{\left(1+\frac{1}{n}\right)^{1/2}-1\right\} = \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+.....\right)-1\right\}$$

$$u_n = \sqrt{n}\left\{\frac{1}{2n}-\frac{1}{8n^2}+...\right\} = \frac{1}{\sqrt{n}}\left\{\frac{1}{2}-\frac{1}{8n}+....\right\}$$

Take $\quad v_n = \dfrac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\sqrt{n}}\left\{\frac{1}{2}-\frac{2}{8n}+......\right\}\div\left(\frac{1}{\sqrt{n}}\right)=\frac{1}{2}$$

which is finite and non-zero .

Using comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\quad \sum v_n = \sum \dfrac{1}{\sqrt{n}}$ is divergent $\left(\text{since } p = 1/2\right)$

$\therefore \quad \sum u_n$ is also divergent.

EXAMPLE 13

Test for convergence $\sum_{n=1}^{\infty}\left[\sqrt[3]{n^3+1}-n\right]$ [JNTU 1996, 2003, 2003]

n^{th} term $\quad u_n = n\left[\left(1+\dfrac{1}{n^3}\right)^{1/3}-1\right] = n\left[1+\dfrac{1}{3n^3}+\dfrac{1/3\left(1/3-1\right)}{1.2}.\dfrac{1}{n^6}+.....-1\right]$

$$= \frac{1}{3n^2}-\frac{1}{9n^5}+......=\frac{1}{n^2}\left(\frac{1}{3}-\frac{1}{9n^3}+......\right) \text{ ; Let } v_n = \frac{1}{n^2}$$

Then $\quad \underset{n\to\infty}{Lt}\, {u_n}/{v_n} = \underset{n\to\infty}{Lt}\left(1/3 - 1/9n^3+....\right) = 1/3 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent by p -series test $\left(\text{since } p = 2 > 1\right)$ $\therefore$ $\sum u_n$ is convergent.

EXAMPLE 14

Show that the series, $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+.......$ is convergent for $p > 2$ and divergent for $p \le 2$

SOLUTION

n^{th} term of the given series = $u_n = \dfrac{n+1}{n^p} = \dfrac{n\left(1+1/n\right)}{n^p} = \dfrac{\left(1+1/n\right)}{n^{p-1}}$

Let us take $v_n = \dfrac{1}{n^{p-1}}$; $\underset{n\to\infty}{Lt}\, {u_n}/{v_n} = 1 \neq 0;$

$\therefore\ \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test.

But $\sum v_n = \sum 1/n^{p-1}$ converges when p -1>1 ; i.e., p >2 and diverges when $p-1 \le 1$ i.e $p \le 2$; Hence the result.

EXAMPLE 15

Test for convergence $\sum_{n=1}^{\infty}\left(\dfrac{2^n+3}{3^n+1}\right)^{1/2}$ (JNTU 2003)

SOLUTION

$$u_n = \left[\frac{2^n\left(1+3/2^n\right)}{3^n\left(1+1/3^n\right)}\right]^{1/2} ; \quad \text{Take} \quad v_n = \sqrt{\frac{2^n}{3^n}} ; \qquad \frac{u_n}{v_n} = \left(\frac{1+3/2^n}{1+1/3^n}\right)^{1/2}$$

$\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = 1 \neq 0$; $\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n = \sum_{n=1}^{\infty}\left(\dfrac{2}{3}\right)^{n/2} = \sqrt{\dfrac{2}{3}} + \dfrac{2}{3} + \left(\dfrac{2}{3}\right)^{3/2} + ...,$ which is a geometric series with common ratio $\sqrt{2/3}$ (<1) $\therefore\ \sum v_n$ is convergent. Hence $\sum u_n$ is convergent.

EXAMPLE 16

Test for convergence of the series, $\dfrac{1}{4.7.10} + \dfrac{4}{7.10.13} + \dfrac{9}{10.13.16} + ...$ (JNTU 2003)

SOLUTION

4, 7, 10,..............is an A . P; $t_n = 4 + (n-1)3 = 3n+1$

7, 10, 13,............is an A . P; $t_n = 7 + (n-1)3 = 3n+4$

and 10 , 13 , 16 ,.............is an A. P; $t_n = 10 + (n-1)3 = 3n+7$

$$\therefore \quad u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{n^2}{3n\left(1+1/3n\right).3n\left(1+4/3n\right).3n\left(1+7/3n\right)}$$

$$= \frac{1}{27n\left(1+1/3n\right)\left(1+4/3n\right)\left(1+7/3n\right)} ;$$

Taking $v_n = \frac{1}{n}$, we get

$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{1}{27} \neq 0$; $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ behave in the same manner. But by p –series test, $\sum v_n$ is divergent, since p = 1. $\therefore$ $\sum u_n$ is divergent .

EXAMPLE 17

Test for convergence $\sum \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$ (JNTU 2003)

SOLUTION

n^{th} term of the given series = $u_n = \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$

Let $v_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \cdot \left[\frac{n\sqrt{2 - 5/n + 1/n^2}}{n^3\left(4 - 7/n + 2/n^3\right)} \times \frac{n^2}{1} \right] = \underset{n\to\infty}{Lt} \left[\frac{\sqrt{2 - 5/n + 1/n^2}}{\left(4 - 7/n + 2/n^3\right)} \right] = \frac{\sqrt{2}}{4} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent. [p series test – p = 2 > 1] $\therefore$ $\sum u_n$ is convergent .

EXAMPLE 18

Test the series $\sum u_n$, whose n^{th} term is $\frac{1}{\left(4n^2 - i\right)}$

SOLUTION

$$u_n = \frac{1}{\left(4n^2 - i\right)}; \qquad \text{Let } v_n = \frac{1}{n^2}, \qquad \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \left[\frac{n^2}{n^2\left(4 - i/n^2\right)} \right] = \frac{1}{4} \neq 0$$

$\therefore$ $\sum u_n$ and $\sum v_n$ both converge or diverge by comparison test. But $\sum v_n$ is convergent by p –series test $\left(p = 2 > 1\right)$; $\therefore$ $\sum u_n$ is convergent.

***Note*:** Test the series $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

EXAMPLE 19

If $u_n = \left(\frac{1}{n}\right).\sin\left(\frac{1}{n}\right)$, show that $\sum u_n$ is convergent .

SOLUTION

Let $v_n = \frac{1}{n^2}$, so that $\sum v_n$ is convergent by p –series test .

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \underset{t\to 0}{Lt}\left(\frac{\sin t}{t}\right)$$

where $t = 1/n$, Thus $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ is convergent.

EXAMPLE 20

Test for convergence $\sum \frac{1}{\sqrt{n}}\tan(1/n)$

SOLUTION

Take $v_n = 1/n^{3/2}$; $\underset{n\to\infty}{Lt}\left[u_n/v_n\right] = 1 \neq 0$ (as in above example)

Hence by comparison test, $\sum u_n$ converges as $\sum v_n$ converges.

EXAMPLE 21

Show that $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ is convergent.

SOLUTION

Let $u_n = \sin^2\left(\frac{1}{n}\right)$; Take $v_n = \frac{1}{n^2}$, $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\sin\left(1/n\right)}{1/n}\right]^2 = \underset{t\to 0}{Lt}\left(\frac{\sin t}{t}\right)^2$

where $t = 1/n$; $\underset{n\to\infty}{Lt}\left(u_n/v_n\right) = 1^2 = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n$ is convergent by p- series test, since p = 2 > 1; $\therefore$ $\sum u_n$ is convergent.

EXAMPLE 22

Show that $\sum_{n=2}^{\infty} 1\big/\log\left(n^n\right)$ is divergent.

SOLUTION

$$u_n = 1\big/n\log n \;;\; \log 2 < 1 \Rightarrow 2\log 2 < 2 \Rightarrow 1\big/2\log 2 > 1\big/2 \;;$$

Similarly $\quad 1\big/3\log 3 > 1\big/3, \ldots\ldots 1\big/n\log n > 1\big/n, n \in N$

$\therefore \quad \sum 1\big/n\log n > \sum 1\big/n$; But $\sum 1\big/n$ is divergent by p-series test.

By comparison test, given series is divergent. [If $\sum v_n$ is divergent and $u_n \geq v_n \forall n$ then $\sum u_n$ is divergent.]

(***Note :*** This problem can also be done using Cauchy's integral Test.

EXAMPLE 23

Test the convergence of the series $\sum_{n=1}^{\infty}(c+n)^{-r}(d+n)^{-s}$, where c, d, r, s are all +ve.

SOLUTION

The n^{th} term of the series $= u_n = \dfrac{1}{(c+n)^r (d+n)^s}$.

Let $\quad v_n = \dfrac{1}{n^{r+s}}$ Then $\dfrac{u_n}{v_n} = \dfrac{n^{r+s}}{n^r\left(1+\dfrac{c}{n}\right)^r .n^s\left(1+\dfrac{d}{n}\right)^s} = \dfrac{1}{\left(1+\dfrac{c}{n}\right)^r\left(1+\dfrac{d}{n}\right)^s}$

$\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = 1 \neq 0$, $\therefore \sum u_n$ and $\sum v_n$ both converge are diverge, by comparison test.

But by *p*-series test, $\sum v_n$ converges if $(r + s) > 1$ and diverges if $(r + s) \leq 1$

$\therefore \sum u_n$ converges if $(r + s) > 1$ and diverges if $(r + s) \leq 1$.

EXAMPLE 24

Show that $\sum_{1}^{\infty} n^{-\left(1+1/n\right)}$ is divergent.

SOLUTION

$$u_n = n^{-\left(1+1/n\right)} = \frac{1}{n.n^{1/n}} \qquad \text{Take} \qquad v_n = \frac{1}{n} \;;\; \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^{1/n}} = 1 \neq 0$$

For let $\underset{n\to\infty}{Lt}\frac{1}{n^{1/n}}=y$ say; log y = $\underset{n\to\infty}{Lt}-\frac{1}{n}.\log n\ =\ -\underset{n\to\infty}{Lt}\frac{1/n}{1}=0$

$\therefore\quad y=e^0=1\qquad$ ($\left(\frac{\infty}{\infty}\right)$ using L Hospitals rule)

By comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge. But *p*-series test, $\sum v_n$ diverges (since p =1); Hence $\sum u_n$ diverges.

EXAMPLE 25

Test for convergence the series $\sum_{n=1}^{\infty}\frac{(n+a)^r}{(n+b)^p(n+c)^q}$, *a, b, c , p, q, r,* being +ve.

SOLUTION

$$u_n=\frac{(n+a)^r}{(n+b)^p(n+c)^q}=\frac{n^r\left(1+a/n\right)^r}{n^p\left(1+b/n\right)^p n^q\left(1+c/n\right)^q}=\frac{1}{n^{p+q-r}}.\frac{\left(1+a/n\right)^r}{\left(1+b/n\right)^p\left(1+c/n\right)^q};$$

Take $v_n=\frac{1}{n^{p+q-r}}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1\neq 0$;

Applying comparison tests both $\sum u_n$ and $\sum v_n$ converge or diverge.

But by *p*-series test, $\sum v_n$ converges if $(p+q-r)>1$ and diverges if $(p+q-r)\leq 1$.

Hence $\sum u_n$ converges if $(p+q-r)>1$ and diverges if $(p+q-r)\leq 1$.

EXAMPLE 26

Test the convergence of the following series whose n^{th} terms are:

(a) $\frac{(3n+4)}{(2n+1)(2n+3)(2n+5)}$; (b) $\tan\frac{1}{n}$; (c) $\left(\frac{1}{n^2}\right)\left(\frac{n+1}{n+3}\right)^n$

(d) $\frac{1}{(3^n+5^n)}$; (e) $\frac{1}{n.3^n}$

SOLUTION

(a) *Hint :* Take $v_n=\frac{1}{n^2};\sum v_n$ is convergent; $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right)=\frac{3}{8}\neq 0$ (Verify)

Apply comparison test:

$\sum u_n$ is convergent [the student is advised to work out this problem fully]

(b) Proceed as in Example 8; $\sum u_n$ is convergent.

(c) *Hint :* Take $v_n = \frac{1}{n^2}$; $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{e}{e^3} = \frac{1}{e^2} \neq 0$

$v_n = \frac{1}{n^2}$ is convergent (work out completely for yourself)

(d) $u_n = \frac{1}{3^n + 5^n} = \frac{1}{5^n} \cdot \frac{1}{\left[1+\left(\frac{3}{5}\right)^n\right]}$; Take $v_n = \frac{1}{5^n}$; $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = 1 \neq 0$

$\sum u_n$ and $\sum v_n$ behave the same way. But $\sum v_n$ is convergent since it is a geometric series with common ratio $\frac{1}{5} < 1$

$\therefore$ $\sum u_n$ is convergent by comparison test .

(e) $\frac{1}{n.3^n} \leq \frac{1}{3^n}, \forall n \in N$, since $n.3^n \geq 3^n$;

$\therefore$ $\sum \frac{1}{n.3^n} \leq \sum \frac{1}{3^n}$ …..(1)

The series on the R.H .S of (1) is convergent since it is geometric series with $r = \frac{1}{3} < 1$.

$\therefore$ By comparison test $\sum \frac{1}{n.3^n}$ is convergent.

EXAMPLE 27

Test the convergence of the following series.

(a) $1 + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \frac{1+2+3+4}{1^2+2^2+3^2+4^2}$ + ...

(b) $1 + \frac{1^2+2^2}{1^3+2^3} + \frac{1^2+2^2+3^2}{1^3+2^3+3^3} + \frac{1^2+2^2+3^2+4^2}{1^3+2^3+3^3+4^3}$ + ...

SOLUTION

(a) $$u_n = \frac{1+2+3+....+n}{1^2+2^2+3^2+.....n^2} = \frac{n\frac{(n+1)}{2}}{n(n+1)\frac{(2n+1)}{6}} = \frac{3}{(2n+1)}$$

Take $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(\frac{3n}{2n+1}\right) = \frac{3}{2} \neq 0$

$\sum u_n$ and $\sum v_n$ behave alike by comparison test.

But $\sum v_n$ is diverges by p-series test. Hence $\sum u_n$ is divergent.

(b) $$u_n = \frac{1^2+2^2+....+n^2}{1^3+2^3+.....n^3} = \frac{n(n+1)\frac{(2n+1)}{6}}{n^2\frac{(n+1)^2}{4}} = \frac{2(2n+1)}{3n(n+1)}$$

Hint : Take $v_n = \frac{1}{n}$ and proceed as in (a) and show that $\sum u_n$ is divergent.

Exercise 1.1

1. Test for convergence the infinite series whose n^{th} term is:

(a) $\frac{1}{n-\sqrt{n}}$ **[Ans :** divergent**]**

(b) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$ **[Ans :** convergent**]**

(c) $\sqrt{n^2+1}-n$ **[Ans :** divergent**]**

(d) $\frac{\sqrt{n}}{n^2-1}$ **[Ans :** convergent**]**

(e) $\sqrt{n^3+1}-\sqrt{n^3}$ **[Ans :** divergent**]**

(f) $\frac{1}{\sqrt{n(n+1)}}$ **[Ans :** divergent**]**

(g) $\frac{\sqrt{n}}{n^2+1}$ **[Ans :** convergent**]**

(h) $\frac{2n^3+5}{4n^5+1}$ **[Ans :** convergent**]**

2. Determine whether the following series are convergent or divergent.

(a) $\frac{1}{1+3^{-1}}+\frac{2}{1+3^{-2}}+\frac{3}{1+3^{-3}}+$ .... **[Ans :** divergent**]**

(b) $\frac{12}{1^3}+\frac{22}{2^3}+\frac{32}{3^3}+........+\frac{2+10n}{n^3}+.....$ .... **[Ans :** convergent**]**

(c) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+......$ .... **[Ans :** divergent**]**

(d) $\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+......$ .... **[Ans :** divergent**]**

(e) $\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^4}+$ .... **[Ans :** convergent**]**

(f) $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n^2+1}}{\sqrt[4]{4n^2+2n+3}}$ .... **[Ans :** divergent**]**

(g) $\sum_{1}^{\infty}\left(8^{1/n}-1\right)$ .... **[Ans :** divergent**]**

(h) $\sum_{1}^{\infty}\frac{3n^3+8}{5n^5+9}$ .... **[Ans :** convergent**]**

(i) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+$ .... **[Ans :** divergent**]**

D' Alembert's Ratio Test

Let (i) $\sum u_n$ be a series of +ve terms and (ii) $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k(\geq 0)$

Then the series $\sum u_n$ is (i) convergent if $k < 1$ and (ii) divergent if $k > 1$.

Proof :

***Case* (i)** $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k(<1)$

From the definition of a limit, it follows that

$\exists m>0$ and $l(0<l<1) \ni \frac{u_{n+1}}{u_n}<l\,\forall n\geq m$

i.e., $\frac{u_{m+1}}{u_m} < l$, $\frac{u_{m+2}}{u_{m+1}} < l$,..........

$$\therefore \quad u_m + u_{m+1} + u_{m+2} + ... = u_m \left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_m} + ...\right]$$

$$u_m \quad 1 \quad \frac{u_{m\ 1}}{u_m} \quad \frac{u_{m\ 2}}{u_{m\ 1}}.\frac{u_{m\ 1}}{u_m} \quad .....$$

$$< u_m \left(1 + l + l^2 + ...\right) = u_m.\frac{1}{1-l}\left(l < 1\right)$$

But $u_m.\frac{1}{1-l}$ is a finite quantity $\therefore \sum_{n=m}^{\infty} u_n$ is convergent

By adding a finite number of terms $u_1 + u_2 + ... + u_{m-1}$, the convergence of the series is unaltered. $\sum_{n=m}^{\infty} u_n$ is convergent.

***Case* (ii)** $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = k > 1$

There may be some finite number of terms in the beginning which do not satisfy the condition $\frac{u_{n+1}}{u_n} \geq 1$. In such a case we can find a number *'m'* $\ni \frac{u_{n+1}}{u_n} \geq 1, \forall n \geq m$

Omitting the first *'m'* terms, if we write the series as $u_1 + u_2 + u_3 +$ ...., we have

$$\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1 \text{ .... and so on}$$

$$\therefore \quad u_1 + u_2 + ... + u_n = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + ...\right) \quad \text{(to } n \text{ terms)}$$

$\geq$ u_1 (1 + 1 + 1.1 +.......to n terms)

= nu_1

$$\underset{n\to\infty}{Lt} \sum_{n=1}^{n} u_n \geq \underset{n\to\infty}{Lt}\ n.u_1 \text{ which } \to \infty; \therefore \sum u_n \text{ is divergent .}$$

***Note*: 1** The ratio test fails when $k = 1$. As an example, consider the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Here $$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^p = \underset{n\to\infty}{Lt} \left(\frac{1}{1+1/n}\right)^p = 1$$

i.e., $k = 1$ for all values of p,

But the series is convergent if $p > 1$ and divergent if $p \le 1$, which shows that when $k = 1$, the series may converge or diverge and hence the test fails .

***Note:* 2** *Ratio test can also be stated as follows:*

If $\sum u_n$ is series of +ve terms and if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then $\sum u_n$ is convergent

If $k > 1$ and divergent if $k < 1$ (the test fails when $k = 1$).

Solved Examples

Test for convergence of Series

EXAMPLE 28

(a) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots\ldots\ldots\ldots\ldots\ldots$ (JNTU 2003)

SOLUTION

$$u_n = \frac{x^n}{n(n+1)};\ u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)};\quad \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n} = \frac{1}{\left(1+\frac{2}{n}\right)} x .$$

Therefore $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x$

$\therefore$ By ratio test $\sum u_n$ is convergent When $|x| < 1$ and divergent when $| x | > 1$;

When $x = 1$, $u_n = \frac{1}{n^2(1+1/n)}$; Take $v_n = \frac{1}{n^2}$; $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test $\sum u_n$ is convergent.

Hence $\sum u_n$ is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

(b) $1+3x+5x^2+7x^3+.......$

SOLUTION

$$u_n=(2n-1)x^{n-1}; \quad u_{n+1}=(2n+1)x^n; \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n-1}\right)x=x$$

$\therefore$ By ratio test $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$

When $x=1: u_n=2n-1;\ \underset{n\to\infty}{Lt}\, u_n=\infty;\ \therefore\ \sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|\geq 1$

(c) $\sum_{n=1}^{\infty}\frac{x^n}{n^2+1}$ ....

SOLUTION

$$u_n=\frac{x^n}{n^2+1}; \quad u_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}.$$

Hence $$\frac{u_{n+1}}{u_n}=\left(\frac{n^2+1}{n^2+2n+2}\right)x, \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\right](x)=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$ When $x=1: u_n=\frac{1}{n^2+1}$; Take $v_n=\frac{1}{n^2}$

$\therefore$ By comparison test, $\sum u_n$ is convergent when $|x|\leq 1$ and divergent when $|x|>1$

EXAMPLE 29

Test the series $\sum_{n\to\infty}^{\infty}\left(\frac{n^2-1}{n^2+1}\right)x^n, x>0$ for convergence.

SOLUTION

$$u_n=\left(\frac{n^2-1}{n^2+1}\right)x^n; u_{n+1}=\left[\frac{(n+1)^2-1}{(n+1)^2+1}\right]x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\left(\frac{n^2+2n}{n^2+2n+2}\right)\frac{(n^2+1)}{(n^2-1)}\right].x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{n^4(1+2/n)(1+1/n^2)}{n^4(1+2/n+2/n^2)(1-1/n^2)}\right]=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$ when $x = 1$,

$$u_n=\frac{n^2-1}{n^2+1}\ \text{Take}\ \ v_n=\frac{1}{n^0}$$

Applying p-series and comparison test, it can be seen that $\sum u_n$ is divergent when $x = 1$.

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent $x \geq 1$

EXAMPLE 30

Show that the series $1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+.....$, is convergent for all values of *p.*

SOLUTION

$$u_n=\frac{n^p}{\lfloor n}\ ;\ u_{n+1}=\frac{(n+1)^p}{\lfloor n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{(n+1)^p}{\lfloor n+1}\times\frac{\lfloor n}{n^p}\right]=\underset{n\to\infty}{Lt}\left\{\frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^p\right\}$$

$$=\underset{n\to\infty}{Lt}\frac{1}{(n+1)}\times\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^p=0<1;$$

$\sum u_n$ is convergent for all ' p ' .

EXAMPLE 31

Test the convergence of the following series

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ ....$$

SOLUTION

$$u_n=\frac{1}{(2n-1)^p};\qquad u_{n+1}=\frac{1}{(2n+1)^p}$$

$$\frac{u_{n+1}}{u_n}=\frac{(2n-1)^p}{(2n+1)^p}=\frac{2^p.n^p\,(1-1/2n)^p}{2^p\,n^p\,(1+1/2n)^p};\qquad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=1$$

$\therefore$ Ratio test fails.

Take $v_n=\frac{1}{n^p}$; $\frac{u_n}{v_n}=\frac{n^p}{(2n-1)^p}=\frac{1}{2^p\left(1-\frac{1}{2n}\right)^p}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{1}{2^p}$,

which is non – zero and finite

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge.

But by p – series test, $\sum v_n = \sum\frac{1}{n^p}$ converges when $p > 1$ and diverges when $p\le 1$

$\therefore$ $\sum u_n$ is convergent if $p > 1$ and divergent if $p\le 1$.

EXAMPLE 32

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(n+1)x^n}{n^3}; x>0$

SOLUTION

$$u_n=\frac{(n+1)x^n}{n^3};u_{n+1}\frac{(n+2)x^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n}=\frac{n+2}{(n+1)^3}.x^{n+1}.\frac{n^3}{(n+1)x^n}=\left(\frac{n+2}{n+1}\right)\left(\frac{n}{n+1}\right)^3.x$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)\frac{1}{\left(1+\frac{1}{n}\right)^3}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, $u_n=\frac{n+1}{n^3}$

Take $v_n=\frac{1}{n^2}$; By comparison test $\sum u_n$ is convergent (give proof)

$\therefore$ $\sum u_n$ is convergent if $x\le 1$ and divergent if $x > 1$.

EXAMPLE 33

Test the convergence of the series (JNTU 2002)

(i) $\sum_{n-1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ (ii) $1+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ ...$ (iii) $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+$

SOLUTION

(i) $\sum_{n-1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{n^2}{2^n}+\sum_{n=1}^{\infty}\frac{1}{n^2}$ Let $u_n=\frac{n^2}{2^n};v_n=\frac{1}{n^2}$

$u_{n+1}=\frac{(n+1)^2}{2^{n+1}};\frac{u_{n+1}}{u_n}=\frac{(n+1)^2}{2^{n+1}}.\frac{2^n}{n^2}$ $\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{1}{2}.\left(1+\frac{1}{n}\right)^2=\frac{1}{2}<1$

$\therefore$ By ratio test $\sum u_n$ is convergent. By p –series test, $\sum v_n$ is convergent.

$\therefore$ The given series $\left(\sum u_n+\sum v_n\right)$ is convergent.

(ii) Neglecting the first term, the series can be taken as, $\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+$

Here, 1st term has 3 fractions ,2nd term has 4 fractions and so on .

$\therefore$ n^{th} term contains (n + 2) fractions

2. 5. 8.......are in A. P.

$\therefore$ $(n+2)^{th}$ term = 2 + (n + 1) 3 = 3n + 5 ;

$\therefore$ 1. 5. 9,.......are in A. P.

$\therefore$ $(n+2)^{th}$ term = 1 + (n + 1) 4 = 4n + 5

$\therefore$ $u_n=\frac{2.5.8.....(3n+5)}{1.5.9.....(4n+5)}$

$u_{n+1}=\frac{2.5.8.....(3n+5)(3n+8)}{1.5.9.....(4n+5)(4n+9)}$

$\frac{u_{n+1}}{u_n}=\frac{(3n+8)}{(4n+9)}$; $\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{n\left(3+\frac{8}{n}\right)}{n\left(4+\frac{9}{n}\right)}=\frac{3}{4}<1$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(iii) 1, 2, 3, ... are in A. P n^{th} term = n ; 3. 5. 7..........are in A.P. n^{th} term = $2n + 1$

$$\therefore \quad u_n = \left[\frac{1.2.3.....n}{3.5.7.....(2n+1)}\right]$$

$$u_{n+1} = \left[\frac{1.2.3.....n(n+1)}{3.5.7.....(2n+1)(2n+3)}\right]$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{2n+3}\right)$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{n.\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = \frac{1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 34

Test for convergence $\sum_{n=1}^{\infty}\frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1}\,(x>0)$ (JNTU 2001)

SOLUTION

The given series of +ve terms has $u_n = \frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1}$

and $u_{n+1} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)}x^n$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n+2}\right)x = \underset{n\to\infty}{Lt}\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{2}{2n}\right)}.x = x$$

$\therefore$ By ratio test, $\sum u_n$ is converges when $x < 1$ and diverges when $x > 1$ when $x = 1$, the test fails.

Then $u_n = \frac{1.3.5....(2n-1)}{2.4.6.....2n} < 1$ and $\underset{n\to\infty}{Lt}\,u_n \neq 0$

$\therefore$ $\sum u_n$ is divergent. Hence $\sum u_n$ is convergent when $x < 1$, and divergent when $x \geq 1$

EXAMPLE 35

Test for the convergence of $1+\frac{2}{5}x+\frac{6}{9}x^2+........+\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}+.....(x>0)$

(JNTU 2003)

SOLUTION

Omitting 1st term, $u_n=\left(\frac{2^n-2}{2^n+1}\right)x^{n-1},(n\geq 2)$ and 'u_n' are all +ve.

$$u_{n+1}=\frac{\left(2^{n+1}-2\right)}{\left(2^{n+1}+1\right)}x^n\ ;\ \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}.\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)\times\left(\frac{2^n+1}{2^n-2}\right).x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1-1/2^n\right)}{2^{n+1}\left(1+1/2^{n+1}\right)}.\frac{2^n\left(1+1/2^n\right)}{2^n\left(1-2/2^n\right)}.x\right]=x\ ;$$

Hence, by ratio test, $\sum u_n$ converges if $x<1$ and diverges if $x>1$.

When $x=1$, the test fails. Then $u_n=\frac{2^n-2}{2^n+1};\ \underset{n\to\infty}{Lt}\ u_n=1\neq 0;\ \therefore\ \sum u_n$ diverges

Hence $\sum u_n$ is convergent when $x<1$ and divergent $x>1$

EXAMPLE 36

Using ratio test show that the series $\sum_{n=0}^{\infty}\frac{(3-4i)^n}{n!}$ converges (JNTU 2000)

SOLUTION

$$u_n=(3-4i)^n/n!\ ;\qquad u_{n+1}=(3-4i)^{n+1}/(n+1)!\ ;\ \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}\left(\frac{3-4i}{n+1}\right)=0<1$$

Hence, by ratio test, $\sum u_n$ converges.

EXAMPLE 37

Discuss the nature of the series, $\frac{2}{3.4}x+\frac{3}{4.5}x^2+\frac{4}{5.6}x^3+........\infty\,(x>0)$ (JNTU 2003)

SOLUTION

Since $x>0$, the series is of +ve terms ;

$$u_n = \frac{(n+1)}{(n+2)(n+3)}x^n > u_{n+1} = \frac{(n+2)}{(n+3)(n+4)}x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \left[\frac{(n+2)^2 .x}{(n+1)(n+4)}\right] = \underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{2}{n}\right)^2 .x}{n^2\left(1+\frac{5}{n}+\frac{4}{n^2}\right)}\right] = x;$$

Therefore by ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$

When $x = 1$, the test fails; Then $u_n = \frac{(n+1)}{(n+2)(n+3)}$;

Taking $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} = \frac{u_n}{v_n} = 1 \neq 0$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ behave same way. But $\sum v_n$ is divergent by *p*-series test. ($p = 1$);

$\therefore$ $\sum u_n$ is diverges when $x = 1$

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

EXAMPLE 38

Discuss the nature of the series $\sum \frac{3.6.9.....3n.5^n}{4.7.10.....(3n+1)(3n+2)}$ (JNTU 2003)

SOLUTION

Here, $u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)}\frac{5^n}{(3n+2)}$;

$$u_{n+1} = \frac{3.6.9.....3n(3n+3)5^{n+1}}{4.7.10.....(3n+1)(3n+4)(3n+5)};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{(3n+2)(3n+3).5}{(3n+4)(3n+5)}$$

$$= \underset{n\to\infty}{Lt}\left[\frac{5.9n^2\left(1+\frac{2}{3n}\right)\left(1+\frac{3}{3n}\right)}{9n^2\left(1+\frac{4}{3n}\right)\left(1+\frac{5}{3n}\right)}\right] = 5 > 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent.

EXAMPLE 39

Test for convergence the series $\sum_{n=1}^{\infty} n^{1-n}$

SOLUTION

$$u_n = n^{1-n};\ u_{n+1} = (n+1)^{-n};$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{-n}}{n^{1-n}} = \frac{n^n}{n(n+1)^n} = \frac{1}{n}\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{1}{n}.\left(\frac{1}{1+1/n}\right)^n = 0.\frac{1}{e} = 0 < 1$$

$\therefore$ By ratio test $\sum u_n$, is convergent

EXAMPLE 40

Test the series $\sum_{n=1}^{\infty}\frac{2n^3}{\lfloor n}$, for convergence.

SOLUTION

$$u_n = \frac{2n^3}{\lfloor n};\ u_{n+1} = \frac{2(n+1)^3}{\lfloor n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)^3}{\lfloor n+1}\times\frac{\lfloor n}{2n^3} = \frac{(n+1)^2}{n^3} = \frac{\left(1+1/n\right)^2}{n};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = 0 < 1;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 41

Test convergence of the series $\sum\frac{2^n n!}{n^n}$

SOLUTION

$$u_n = \frac{2^n n!}{n^n};\ u_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}};$$

$$\frac{u_{n+1}}{u_n}=\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!}=2\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=2\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{2}{e}<1 \quad \text{(since } 2<e<3\text{)}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 42

Test the convergence of the series $\sum u_n$ where u_n is

(a) $\dfrac{n^2+1}{3^n+1}$ (b) $\dfrac{x^{n-1}}{(2n+1)^a},(a>0)$ (c) $\left(\dfrac{1.2.3....n}{4.7.10.....3n+3}\right)^2$

(d) $\dfrac{\sqrt{1+2^n}}{\sqrt{1+3^n}}$ (e) $\left(\dfrac{3n^3+7n^2}{5n^9+11}\right)x^n$

SOLUTION

(a) $$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}\left[\frac{(n+1)^2+1}{3^{n+1}+1}\times\frac{3^n+1}{n^2+1}\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}\cdot\frac{3^n\left(1+\frac{1}{3^n}\right)}{3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)}\right]$$

$$=\frac{1}{3}<1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(b) $$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}\left[\frac{x^n}{(2n+3)^a}\times\frac{(2n+1)^a}{x^{n-1}}\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{2^a n^a\left(1+\frac{1}{2n}\right)^a}{2^a n^a\left(1+\frac{3}{2n}\right)^a}.x\right]=x$$

By ratio test, $\sum u_n$ convergence if $x<1$ and diverges if $x>1$.

When $x = 1$, the test fails; Then, $u_n = \frac{1}{(2n+1)^a}$; Taking $v_n = \frac{1}{n^a}$ we have,

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right)^a = \underset{n\to\infty}{Lt}\frac{1}{\left(2+\frac{1}{n}\right)^a} = \frac{1}{2^a} \neq 0 \text{ and finite } (\text{since } a > 0).$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ have same property

But p –series test, we have

(i) $\sum v_n$ convergent when $a > 1$

and (ii) divergent when $a \le 1$

$\therefore$ To sum up, (i) $x < 1$, $\sum u_n$ is convergent $\forall a$.

(ii) $x > 1$, $\sum u_n$ is divergent $\forall a$.

(iii) $x = 1$, $a > 1$, $\sum u_n$ is convergent, and

(iv) $x = 1$, $a \le 1$, $\sum u_n$ is divergent.

(c) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{1.2.3....n(n+1)}{4.7.10....(3n+3)(3n+6)} \times \frac{4.7.10....(3n+3)}{1.2.3....n}\right]^2$$

$$= \underset{n\to\infty}{Lt}\left[\frac{(n+1)}{3(n+2)}\right]^2 = \frac{1}{9} < 1 \;;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent

(d) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(1+2^{n+1})}{(1+3^{n+1})} \times \frac{(1+3^n)}{(1+2^n)}\right]^{1/2}$$

$$= \underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}{3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)} \times \frac{3^n\left(1+\frac{1}{3^n}\right)}{2^n\left(1+\frac{1}{2^n}\right)}\right]^{1/2} = \left(\frac{2}{3}\right)^{1/2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(e) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{3(n+1)^3+7(n+1)^2}{5(n+1)^9+11}\times\frac{5n^9+11}{3n^3+7}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left(1+\frac{1}{n}\right)^3+7n^2\left(1+\frac{1}{n}\right)^2}{5n^9\left(1+\frac{1}{n}\right)^9+11}\times\frac{5n^9\left(1+\frac{11}{5n^9}\right)}{3n^3\left(1+\frac{7}{3n^3}\right)}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left\{\left(1+\frac{1}{n}\right)^3+\frac{7}{3n}\left(1+\frac{1}{n}\right)^2\right\}}{5n^9\left\{\left(1+\frac{1}{n}\right)^9+\frac{11}{5n^9}\right\}}\times\frac{5n^9\left(1+\frac{11}{5n^9}\right)}{3n^3\left(1+\frac{7}{3n^3}\right)}\times x\right]=x$$

∴ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, the test fails,

Then $$u_n=\frac{3n^3\left(1+\frac{7}{3n}\right)}{5n^9\left(1+\frac{11}{5n^9}\right)}=\frac{3}{5n^6}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{11}{5n^9}\right)}$$

Taking $v_n=\dfrac{1}{n^6}$, we observe that $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}=\dfrac{3}{5}\neq 0$

∴ By comparison test and p series test, we conclude that $\sum u_n$ is convergent.

∴ $\sum u_n$ is convergent when $x\le 1$ and divergent when $x > 1$.

Exercise – 1.2

1. Test the convergency or divergency of the series whose general term is :

(a) $\dfrac{x^n}{n}$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(b) nx^{n-1} .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(c) $\left(\dfrac{2^n-2}{2^n+1}\right)x^{n-1}$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(d) $\left(\dfrac{n^2+1}{n^2-1}\right)x^n$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(e) $\dfrac{\lfloor n}{n^n}$ .... **[Ans: cgt.]**

(f) $\frac{4^n \lfloor n}{n^n}$ .... **[Ans: dgt.]**

(g) $\frac{(n^3+1)^n}{(3^n+1)}$ .... **[Ans: cgt.]**

2. Determine whether the following series are convergent or divergent :

(a) $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+$ .... **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

(b) $1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+$............. **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

(c) $\frac{1}{1.2.3}+\frac{x}{4.5.6}+\frac{x^2}{7.8.9}+$ ...... **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

(d) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+.....\frac{x^n}{n^2+1}+....$ **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

(e) $\frac{1.2}{x}+\frac{2.3}{x^2}+\frac{3.4}{x^3}+$ .... **[Ans :** $|x|>1 cgt, |x|\le 1 dgt$ **]**

Raabe's Test

Let $\sum u_n$ be series of +ve terms and let $\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=k$

Then

(i) If $k > 1$, $\sum u_n$ is convergent. **(ii)** If $k < 1$, $\sum u_n$ is divergent. (The test fails if $k = 1$)

Proof : Consider the series $\sum v_n = \sum \frac{1}{n^p}$

$$n\left[\frac{v_n}{v_{n+1}}-1\right]=n\left[\left(\frac{n+1}{n}\right)^p-1\right]=n\left[\left(1+\frac{1}{n}\right)^p-1\right]$$

$$=n\left[\left(1+\frac{p}{n}+\frac{p(p-1)}{\lfloor 2}.\frac{1}{n^2}+....\right)-1\right]$$

$$= p+\frac{p(p-1)}{\lfloor 2}.\frac{1}{n}+ \text{..........} \qquad \underset{n\to\infty}{Lt}\, n\left\{\frac{v_n}{v_{n+1}}-1\right\}=p$$

***Case* (i)** In this case, $\underset{n\to\infty}{Lt}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k>1$

We choose a number 'p' $\ni k > p > 1$; Comparing the series $\sum u_n$ with $\sum v_n$ which is convergent , we get that $\sum u_n$ will converge if after some fixed number of terms

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n}\right)^p$$

i.e. If, $n\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p-1)}{\angle 2}.\frac{1}{n}+$ ....from (1)

i.e., If $\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right) > p$

i.e., If $k > p$, which is true . Hence $\sum u_n$ is convergent .The second case also can be proved similarly.

Solved Examples

EXAMPLE 43

Test for convergence the series

$$x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$ (JNTU 2006, 2008)

SOLUTION

Neglecting the first tem ,the series can be taken as ,

$$\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$

1.3.5....are in A.P. n^{th} term = $1+(n-1)2=2n-1$

2.4.6...are in A.p. n^{th} term = $2+(n-1)2=2n$

3.5.7.....are in A.P n^{th} term = $3+(n-1)2=2n+1$

$\therefore$ u_n (n^{th} term of the series) $=\frac{1.3.5....(2n-1)}{2.4.6....(2n)}.\frac{x^{2n+1}}{2n+1}$

$$u_{n+1} = \frac{1.3.5....(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)} . \frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)} . \frac{x^{2n+3}}{(2n+3)} . \frac{2.4.6....2n}{1.3.5....(2n-1)} . \frac{(2n+1)}{x^{2n+1}}$$

$$= \frac{(2n+1)^2 x^2}{(2n+2)(2n+3)}$$

$$\therefore \qquad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)} x^2 = x^2$$

$\therefore$ By ratio test, $\sum u_n$ converges if $|x| < 1$ and diverges if $|x| > 1$

If $|x| = 1$ the test fails.

Then $\qquad x^2 = 1$ and $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 = \frac{6n+5}{(2n+1)^2}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{6n^2+5n}{4n^2+4n+1}\right)$$

$$= \underset{n\to\infty}{Lt} \frac{n^2\left(6+\frac{5}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)} = \frac{3}{2} > 1$$

By Raabe's test, $\sum u_n$ converges. Hence the given series is convergent when $|x| \le 1$ an divergent when $|x| > 1$.

EXAMPLE 44

Test for the convergence of the series (JNTU 2007)

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + ...; x > 0$$

SOLUTION

Neglecting the first term,

$$u_n = \frac{3.6.9....3n}{7.10.13....3n+4}.x^n$$

$$u_{n+1} = \frac{3.6.9....3n(3n+3)}{7.10.13....(3n+4)(3n+7)}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}.x \;\; ; \;\; \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x < 1$ and divergent when $x > 1$.

When $x = 1$ The ratio test fails. Then

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}; \frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

$$\underset{n\to\infty}{Lt} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \underset{n\to\infty}{Lt} \left(\frac{4n}{3n+3}\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent .Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

EXAMPLE 45

Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}$

SOLUTION

$$u_n = \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}; \qquad u_{n+1} = \frac{1^2.5^2.9^2....(4n-3)^2(4n+1)^2}{4^2.8^2.12^2....(4n)^2(4n+4)^2}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{(4n+1)^2}{(4n+4)^2} = 1 \;\; \text{(verify)}$$

$\therefore$ The ratio test fails. Hence by Raabe's test, $\sum u_n$ is convergent. (give proof)

EXAMPLE 46

Find the nature of the series $\sum \frac{\left(\lfloor n\right)^2}{\lfloor 2n} x^n, \left(x>0\right)$ (JNTU 2003)

SOLUTION

$$u_n = \frac{\left(\lfloor n\right)^2}{\lfloor 2n}.x^n \ ; \ u_{n+1} = \frac{\left(\lfloor n+1\right)^2}{\lfloor 2n+2}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{\left(n+1\right)^2}{\left(2n+1\right)\left(2n+2\right)}x\,;$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}.x = \frac{x}{4}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{x}{4}<1$, i. e ; x < 4; and diverges when x >4;

When x = 4, the test fails.

$$\frac{u_n}{u_{n+1}} = \frac{\left(2n+1\right)\left(2n+2\right)}{4\left(n+1\right)^2}$$

$$\frac{u_n}{u_{n+1}}-1 = \frac{-2n-2}{4\left(n+1\right)^2} = \frac{-1}{2\left(n+1\right)}\,; \quad \underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \frac{-1}{2}<1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent

Hence $\sum u_n$ is convergent when x < 4 and divergent when $x \geq 4$

EXAMPLE 47

Test for convergence of the series $\sum \frac{4.7....\left(3n+1\right)}{1.2.3....n}x^n$ (JNTU 1996)

SOLUTION

$$u_n = \frac{4.7....\left(3n+1\right)}{1.2.3....n}x^n \ ; \ u_{n+1} = \frac{4.7....\left(3n+1\right)\left(3n+4\right)}{1.2.3....n\left(n+1\right).}x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{\left(3n+4\right)}{\left(n+1\right)}.x\right] = 3x$$

$\therefore$ By ratio test $\sum u_n$ converges if $3x < 1$ i.e., $x < \frac{1}{3}$ and diverges if $x > \frac{1}{3}$;

If $x = \frac{1}{3}$, the test fails

When $x = \frac{1}{3}$, $n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{(n+1)3}{3n+4} - 1\right] = n\left[\frac{-1}{3n+4}\right] = -\frac{1}{\left(3 + \frac{4}{n}\right)}$

$$\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}} - 1\right] = -\frac{1}{3} < 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is divergent.

$\therefore$ $\sum u_n$ is convergent when $x < \frac{1}{3}$ and divergent when $x \geq \frac{1}{3}$

EXAMPLE 48

Test for convergence $2 + \frac{3x}{2} + \frac{4x^2}{3} + \frac{5x^3}{4} + ...(x > 0)$ (JNTU 2003)

SOLUTION

The n^{th} term $u_n = \frac{(n+1)}{n} x^{n-1}$; $u_{n+1} = \frac{(n+2)}{(n+1)} x^n$; $\frac{u_{n+1}}{u_n} = \frac{n(n+2)}{(n+1)^2} .x$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n^2\left(1 + 2/n\right)}{n^2\left(1 + 1/n\right)^2} .x = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $x < 1$ and divergent if $x > 1$

If $x = 1$, the test fails.

Then $\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}} - 1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{(n+1)^2}{n(n+2)} - 1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{1}{n(n+2)}\right] = 0 < 1$

$\therefore$ By Raabe's test $\sum u_n$ is divergent

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

EXAMPLE 49

Find the nature of the series $\frac{3}{4} + \frac{3.6}{4.7} + \frac{3.6.9}{4.7.10} + ...\infty$ (JNTU 2003)

SOLUTION

$$u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)}\ ;\ u_{n+1} = \frac{3.6.9.....3n(3n+3)}{4.7.10.....(3n+1)(3n+4)}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+4};\ \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{3n\left(1+\frac{3}{3n}\right)}{3n\left(1+\frac{4}{3n}\right)} = 1$$

Ratio test fails.

$$\therefore \qquad \underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right] = \underset{n\to\infty}{Lt}\left[n\left(\frac{3n+4}{3n+3}-1\right)\right]$$

$$= \underset{n\to\infty}{Lt}\frac{n}{3(n+1)} = \underset{n\to\infty}{Lt}\frac{n}{3n\left(1+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent.

EXAMPLE 50

If *p, q > 0* and the series

$$1+\frac{1}{2}\frac{p}{q}+\frac{1.3.p(p+1)}{2.4.q(q+1)}+\frac{1.3.5}{2.4.6}\frac{p(p+1)(p+2)}{q(q+1)(q+2)}+....$$

is convergent , find the relation to be satisfied by p and q.

SOLUTION

$$u_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n}\frac{p(p+1).....(p+n-1)}{q(q+1).....(q+n-1)} \text{ [neglecting 1}^{\text{st}}\text{ term]}$$

$$u_{n+1} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6.....2n(2n+2)}\frac{p(p+1).....(p+n-1)(p+n)}{q(q+1).....(q+n-1)(q+n)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)}{(2n+2)}\frac{(p+n)}{(q+n)};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)}.\frac{n\left(1+\frac{p}{n}\right)}{n\left(1+\frac{q}{n}\right)}\right] = 1$$

$\therefore$ ratio test fails.

Let us apply Raabe's test

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=\underset{n\to\infty}{Lt}\left[n\left\{\frac{(q+n)(2n+2)}{(p+n)(2n+1)}-1\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[n\left\{\frac{2q(n+1)-p(2n+1)+n}{n^2\left(1+p/n\right)\left(2+1/n\right)}\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[\frac{2q\left(1+1/n\right)-p\left(2+1/n\right)+1}{2}\right]=\frac{2q-2p+1}{2}$$

Since $\sum u_n$ is convergent, by Raabe's test, $\frac{2q-2p+1}{2}>1$

$\Rightarrow q-p>1/2$, is the required relation.

Exercise 1.3

1. Test whether the series $\sum_1^\infty u_n$ is convergent or divergent where

$$u_n=\frac{2^2.4^2.6^2.....(2n-2)^2}{3.4.5.....(2n-1)(2n)}.x^{2n}$$ **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

2. Test for the convergence the series

$$\sum_1^\infty\frac{4.7.10.....(3n+1)}{\lfloor n}x^n$$ **[Ans :** $|x|<\frac{1}{3}cgt, |x|\ge\frac{1}{3}dgt$ **]**

3. Test for the convergence the series :

(i) $\frac{2^2.4^2}{3^2.3^2}+\frac{2^2.4^2.5^2.7^2}{3^2.3^2.6^2.6^2}+\frac{2^2.4^2.5^2.7^2.8^2.10^2}{3^2.3^2.6^2.6^2.9^2.9^2}+...$ **[Ans** : divergent**]**

(ii) $\frac{3.4}{1.2}x+\frac{4.5}{2.3}x^2+\frac{5.6}{3.4}x^3+.....(x>0)$ **[Ans :** cgt if $x \le 1$ dgt if , $x > 1$**]**

(iii) $\sum\frac{1.3.5.....(2n-1)}{2.4.6.....2n}.\frac{x^n}{(2n+2)}(x>0)$ **[Ans :** cgt if $x \le 1$ dgt if , $x > 1$**]**

(iv) $1+\frac{(\lfloor 1)^2}{\lfloor 2}x+\frac{(\lfloor 2)^2 x^2}{\lfloor 4}+\frac{(\lfloor 3)^2 x^3}{\lfloor 6}+......(x>0)$

[Ans : cgt if $x<4$ and dgt if , $x\geq 4$ **]**

1.3.5 Cauchy's Root Test

Let $\sum u_n$ be a series of +ve terms and let $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l$. Then $\sum u_n$ is convergent when $l < 1$ and divergent when $l > 1$

***Proof* :** (i) $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l<1 \Rightarrow \exists a$ +ve number $'\lambda'(l<\lambda<1) \ni u_n^{1/n}<\lambda, \forall n>m$

(or) $u_n<\lambda^n, \forall n>m$

Since $\lambda<1, \sum\lambda^n$ is a geometric series with common ratio < 1 and therefore convergent.

Hence $\sum u_n$ is convergent.

(ii) $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l>1$

$\therefore$ By the definition of a limit we can find a number $r \ni u_n^{1/n}>1 \forall n>r$

i.e., $u_n > \forall n>r$

i.e., after the 1st 'r' terms , each term is > 1.

$\underset{n\to\infty}{Lt}\sum u_n=\infty \quad \therefore \quad \sum u_n$ is divergent.

Note : When $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right)=1,$ the root test can't decide the nature of $\sum u_n$. The fact of this statement can be observed by the following two examples.

1. Consider the series $\sum 1/n^3 :-\underset{n\to\infty}{Lt}u_n^{1/n}=\underset{n\to\infty}{Lt}\left(\frac{1}{n^3}\right)^{1/n}=\underset{n\to\infty}{Lt}\left(\frac{1}{n^{1/n}}\right)^3=1$

2. Consider the series $\sum 1/n,$ in which $\underset{n\to\infty}{Lt}u_n^{1/n}=\underset{n\to\infty}{Lt}\frac{1}{n^{1/n}}=1$

In both the examples given above**,** $\underset{n\to\infty}{Lt}u_n^{1/n}=1$. But series (1) is convergent (p-series test)

And series (2) is divergent. Hence when the *limit*=1, the test fails.

Solved Examples

EXAMPLE 51

Test for convergence the infinite series whose n^{th} terms are:

(i) $\dfrac{1}{n^{2n}}$ **(ii)** $\dfrac{1}{(\log n)^n}$ **(iii)** $\dfrac{1}{\left[1+\dfrac{1}{n}\right]^{n^2}}$ (JNTU 1996, 1998, 2001)

SOLUTION

(i) $u_n = \dfrac{1}{n^{2n}}, u_n^{1/n} = \dfrac{1}{n^2}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \dfrac{1}{n^2} = 0 < 1;$

By root test $\sum u_n$ is convergent.

(ii) $u_n = \dfrac{1}{(\log n)^n}; u_n^{1/n} = \dfrac{1}{\log n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \dfrac{1}{\log n} = 0 < 1;$

$\therefore$ By root test , $\sum u_n$ is convergent.

(iii) $u_n = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n^2}}; u_n^{1/n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n}}$ $\underset{n\to\infty}{Lt} u_n^{1/n} = \underset{n\to\infty}{Lt} \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n}} = \dfrac{1}{e} < 1;$

$\therefore$ By root test $\sum u_n$ is convergent.

EXAMPLE 52

Find whether the following series are convergent or divergent.

(i) $\sum_{n=1}^{\infty} \dfrac{1}{3^n - 1}$ (ii) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^3} + \dfrac{1}{4^4} + ...$ (iii) $\sum_{n=1}^{\infty} \dfrac{\left[(n+1)x\right]^n}{n^{n+1}}$

SOLUTION

(i) $u_n^{1/n} = \left(\dfrac{1}{3^n - 1}\right)^{1/n} = \left(\dfrac{1}{3^n\left(1-\dfrac{1}{3^n}\right)}\right)^{1/n}$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{3^n\left(1-\frac{1}{3^n}\right)}\right)^{1/n} = \frac{1}{3} < 1;$$ By root test, $\sum u_n$ is convergent.

(ii) $u_n = \frac{1}{n^n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^n}\right)^{1/n} = 0 < 1$; By root test, $\sum u_n$ is convergent.

(iii) $u_n = \frac{\left[(n+1)x\right]^n}{n^{n+1}}$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{\{(n+1)x\}^n}{n^{n+1}}\right]^{1/n}$$

$$\underset{n\to\infty}{Lt}\left[\left\{\frac{(n+1)x}{n}\right\}^n.\frac{1}{n}\right]^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n}\right)x.\frac{1}{n^{1/n}}$$

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)x.\frac{1}{n^{1/n}} = \underset{n\to\infty}{Lt}\, x.\frac{1}{n^{1/n}} = x \qquad \left(\text{since } \underset{n\to\infty}{Lt}\, x.\frac{1}{n^{1/n}} = 1\right)$$

$\therefore$ $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$ and when $|x| = 1$ the test fails.

Then $u_n = \frac{(n+1)^n}{n^{n+1}}$; Take $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}}.n = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n; \quad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = e > 1$$

$\therefore$ By comparison test, $\sum u_n$ is divergent.

($\sum v_n$ diverges by p –series test)

Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent $|x| \geq 1$

EXAMPLE 53

If $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$, show that $\sum u_n$ is convergent.

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} \; ; = \underset{n\to\infty}{Lt} = \frac{n^n}{(n+1)^n} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^n$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 \; ; \; \therefore \; \sum u_n \text{ converges by root test .}$$

EXAMPLE 54

Establish the convergence of the series $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \, ...$

SOLUTION

$$u_n = \left(\frac{n}{2n+1}\right)^n \text{ ....(verify);} \qquad \underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

By root test, $\sum u_n$ is convergent.

EXAMPLE 55

Test for the convergence of $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}.x^n$

SOLUTION

$$u_n = \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x^n \, ; \; \underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x = x$$

$\therefore$ By root test, $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$.

When $|x| = 1$: $u_n = \sqrt{\frac{n}{n+1}}$, taking $v_n = \frac{1}{n^0}$ and applying comparison test , it can be seen that is divergent

$\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

EXAMPLE 56

Show that $\sum_{n=1}^{\infty}\left(n^{1/n} - 1\right)^n$ converges.

SOLUTION

$$u_n = \left(n^{1/n} - 1\right)^n$$

$$\underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left(n^{1/n} - 1\right) = 1 - 1 = 0 < 1\left(\text{since } \underset{n\to\infty}{Lt}\ n^{1/n} = 1\right);$$

$\therefore$ $\sum u_n$ is convergent by root test.

EXAMPLE 57

Examine the convergence of the series whose n^{th} term is $\left(\dfrac{n+2}{n+3}\right)^n .x^n$

SOLUTION

$$u_n = \left(\frac{n+2}{n+3}\right)^n .x^n;\ \underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n+2}{n+3}\right)x = x$$

$\therefore$ By root test, $\sum u_n$ converges when $|x| < 1$ and diverges when $|x| > 1$.

When $|x| = 1$: $u_n = \left(\dfrac{n+2}{n+3}\right)^n$; $\underset{n\to\infty}{Lt}\ u_n = \underset{n\to\infty}{Lt}\dfrac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$

$$= \frac{e^2}{e^3} = \frac{1}{e} \neq 0 \quad \text{and the terms are all +ve .}$$

$\therefore$ $\sum u_n$ is divergent . Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

EXAMPLE 58

Show that the series,

$$\left[\frac{2^2}{1^2} - \frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3} - \frac{3}{2}\right]^{-2} + \left[\frac{4^4}{3^4} - \frac{4}{3}\right]^{-3} + \ldots\ldots \text{ is convergent} \qquad \text{(JNTU 2002)}$$

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}; = \left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-n}$$

$$\left(1+\frac{1}{n}\right)^{-n}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-n};\ u_n^{1/n} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)} \frac{1}{\left\{\left(1+\frac{1}{n}\right)^{n} - 1\right\}}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\ u_n^{1/n} = \frac{1}{1}\cdot\frac{1}{e-1} = \frac{1}{e-1} < 1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

EXAMPLE 59

Test $\sum_{m=1}^{\infty} u_m$ for convergence when $u_m = \dfrac{e^{-m}}{\left(1+2/m\right)^{-m^2}}$

SOLUTION

$$\underset{m\to\infty}{Lt}\left(u_m^{1/m}\right) = \underset{m\to\infty}{Lt}\left[\frac{\left(1+2/m\right)^{m^2}}{e^m}\right]^{1/m} \quad ; \quad \underset{m\to\infty}{Lt}\ \frac{1}{e}\left(1+\frac{2}{m}\right)^m = \frac{e^2}{e} = e > 1$$

Hence Cauchy's root tells us that $\sum u_m$ is divergent.

EXAMPLE 60

Test the convergence of the series $\sum \dfrac{n}{e^{n^2}}$.

SOLUTION

$$\underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\ \frac{n^{1/n}}{e^n} = 0 < 1 \qquad \therefore \text{ By root test, } \sum u_n \text{ is convergent.}$$

EXAMPLE 61

Test the convergence of the series, $\dfrac{2}{1^2}x + \dfrac{3^2}{2^3}x^2 + \ldots \dfrac{(n+1)^n .x^n}{n^{n+1}} + \ldots\ldots, x > 0$

SOLUTION

$$\underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^n .x^n}{n^{n+1}}\right]^{1/n} = \underset{n\to\infty}{Lt}\left[\left(\frac{n+1}{n}\right)\cdot\frac{1}{n^{1/n}}.x\right]$$

$$= \underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right).\frac{1}{n^{1/n}}.x\right] = 1.1.x = x\left[\text{since } \underset{n\to\infty}{Lt}\, n^{1/n} = 1\right]$$

$\therefore$ By root test, $\sum u_n$ converges if $x<1$ and diverges when $x>1$.

When $x=1$, the test fails.

Then $\quad u_n = \left(1+\frac{1}{n}\right)^n.\frac{1}{n}$; Take $v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ By comparison test and *p*-series test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $x<1$ and divergent when $x \geq 1$.

Exercise 1.4

1. Test for convergence the infinite series whose n^{th} terms are:

(a) $\frac{1}{2^n-1}$ .... **[Ans :** convergent**]**

(b) $\frac{1}{(\log)^{2n}}.(n\neq 1)$ .... **[Ans :** convergent**]**

(c) $\left(\frac{3n+1}{4n+3}.x\right)^n$ .... **[Ans :** $|x|<\frac{4}{3}cgt, |x|\geq\frac{4}{3}dgt$ **]**

(d) $\frac{x^n}{\lfloor n}$ .... **[Ans :** cgt for all $x \geq 0$ **]**

(e) $\frac{\lfloor n}{n^n}$ .... **[Ans :** convergent**]**

(f) $\frac{3^n.\angle n}{n^3}$ .... **[Ans :** convergent**]**

(g) $\frac{(2n^2-1)^n}{(2n)^{2n}}$ .... **[Ans :** convergent**]**

(h) $\left(n^{1/n}-1\right)^{2n}$ .... **[Ans :** convergent**]**